M. Math Mid-Semester 2010 Exam Functional Analysis

Answer all the questions. Each question is worth 5 points. You may state correctly and use any result proved in the class. However if an answer is an almost immediate consequence of the stated result, such a result also need to be proved.

All topological spaces are assumed to be Hausdorff.

1) Let X be a complex normed linear space. Let $f : X \to C$ be a non-zero linear map . Show that either $f(B(0,1))$ is a bounded set or all of C. In the second case show that $ker(f)$ is dense in X.

2) Show that for any normed linear space X, X^* is a Banach space.

3) Let $M = \{f \in C([0,1]) : f([0, \frac{1}{2}]) = 0\}$. Let $\Phi : C([0,1])|M \to C([0, \frac{1}{2}])$ be defined by $\phi([f]) = f|[0, \frac{1}{2}]$. Show that Φ is a well-defined, liner, onto, isometry.

4) Let X be a normed linear space and M a closed subspace. Let $\pi : X \to X|M$ be the quotient map. Show that $||\pi|| = 1$.

5) Let H be a complex separable Hilbert space. Show that for some discrete set Δ, there is a linear, continuous, onto map from $H \to \ell^2(\Delta)$.

6) Let H be a complex Hilbert space. Let $P : H \to H$ be a linear map such that $P(P(x)) = P(x)$ and $||P(x)||^2 + ||x - P(x)||^2 = ||x||^2$ for all x. Show that $||P|| = 1 = ||I - P||$, where I denotes the identity map.

7) Let $(\Omega, \mathcal{A}, P)$ be a probability space . Let $\{f_n\}_{n \geq 1} \subset L^3(P)$ be a sequence such that $f_n \to f$ for some $f \in L^3(P)$. Show that for any $g \in L^{\frac{3}{2}}(P)$, $\int f_n g \, dP \to \int fg \, dP$.

8) Let $(X, \mathcal{T})$ be a locally compact non-compact space. Give complete details to show that $C_0(X)$ is a Banach space with the supremum norm.